The Polyphemus Perspective: Uses of Single Factor Models

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R-Squared Risk Management

A Few Words on this Talk …

- We will consider 3 single factor risk models

- The single factor in each of these models will be :-
    - The Market
    - Some Benchmark
    - Your Portfolio

- Each of these simple models has its uses for portfolio managers …

1. The Market Model

- The **Market Model** was the first risk model ever proposed (by Bill Sharpe) (and Jack Treynor)
- Its single explanatory factor is the Market :-

$$R_{it} = \beta_{iM} R_{Mt} + \alpha_{it}$$

- The model says that all stocks go up and down to some extent with the Market, (which is fair enough) and that everything else is independent - **NOT!**

Stock Risk

- Risk is defined as the Variance of returns :-

$$V_i = Var(R_{it}) = Var(\beta_{iM} R_{Mt} + \alpha_{it})$$

$$= \beta_{iM}{}^2 V_M + Var(\alpha_{it})$$

$$= \beta_{iM}{}^2 V_M + RSD_i{}^2$$

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Risk Terminology

- The risk of a single stock is therefore given by :-

$$V_i = \beta_{iM}{}^2 V_M + RSD_i^2$$

- These are called Systematic Risk and Residual Risk
- Systematic Risk is the (Market) factor-related risk
- Residual Risk (by construction) is the part of stock risk that is independent of the (Market) factor
- This distinction between factor-related (Systematic) and factor-independent (Residual) will be useful later

Relationship of Stock to Market

- The return and risk of stock i therefore consists of two parts, a Market-related return and a Residual return

- The relationship with the Market return is given by :-

$$COV_{iM} = COV(R_{it}, R_{Mt}) = COV(\beta_{iM} R_{Mt} + \alpha_{it}, R_{Mt})$$

$$= COV(\beta_{iM} R_{Mt}, R_{Mt}) + COV(\alpha_{it}, R_{Mt})$$

So $COV_{iM} = \beta_{iM} V_M$ since $COV(\alpha_{it}, R_{Mt}) = 0$

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Definition of Stock Beta

- From this, we can derive the usual expression for the Beta of stock i to the (single) Market factor :-

$$\beta_{iM} = \frac{COV_{iM}}{V_M}$$

- We will see variations of this expression as we go through the different single factor models

Covariance between Two Stocks

The Covariance between two stocks is given by :-

$$COV_{ij} = \beta_{iM}\beta_{jM}V_M$$

Since we assume :-

$$COV(\alpha_{it}, \alpha_{jt}) = 0$$

- which is not actually true, of course

Portfolio Risk

- Using this model, portfolio risk becomes :-

$$V_P = \beta_{PM}{}^2 V_M + RSD_P^2$$

where $$\beta_{PM} = \sum_{i=1}^{N} x_i \beta_{iM}$$

and $$RSD_P{}^2 = \sum_{I=1}^{N} x_i{}^2 RSD_i{}^2$$

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The Meaning of Alpha

- In the usual expression of the Market Model,

$$\alpha_{it} = \alpha_i + \varepsilon_{it}$$

- Traditionally, the Residual Return each period is separated into a constant term α_i and an 'error' term ε_{it} with a time-series mean of zero

- Note, however, that this does not stop portfolio managers claiming all the Residual Return in each period as their 'Alpha' for that period

Summarising the Market Model

- This is mainly useful as an introduction to single factor models, and the idea of Beta

- It separates both stock and portfolio risk and return into Systematic (factor-related) and Residual (factor-independent) parts

- Market Betas are assumed to be reasonably stable over time, and are usually estimated by time-series regressions

The Use of the CAPM …

- Finance academics still spend their time testing whether the CAPM holds (or not)

- Portfolio managers know perfectly well that it doesn't – there are many more common factor influences on a stock than just the Market Beta

- In fact, most active portfolio managers these days use some form of multi-factor model to select stocks to hold in their portfolios …

... and Abuse of the CAPM

- Any sensible multi-factor model will ‘explain’ more of a portfolio’s returns than a single-factor model
- As more of the performance of a portfolio is attributed to Systematic (factor) bets, there will be less attributable to Residual Return, or Alpha
- So fund managers typically switch to using the CAPM when they need to make their Alpha as big as possible … *Caveat Investor!*

2. The 'Some Benchmark' Model

- In this model, **Some Benchmark** is the single factor, and we model the returns to stocks and the portfolio by their relationship with this (arbitrary) Benchmark

- The model is as follows :-

$$R_{it} = \beta_{iB} R_{Bt} + \alpha_{it}$$

- Nearly all risk analysis systems provide the Beta of Your Portfolio to the Benchmark, despite the fact that the Benchmark is rarely a factor in the risk model

Portfolio Beta to the Benchmark - 1

- Providing this Beta allows the manager to think of the Portfolio Risk as consisting of two parts :-

  - Systematic Risk – linked to the Benchmark
  - Residual Risk – independent of the Benchmark

- The link to the Benchmark is expressed by the $Beta_{PB}$, which tells us whether the Portfolio returns tend to exaggerate or dampen the returns to the Benchmark

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Portfolio Beta to the Benchmark - 2

- This Beta is also calculated in the usual way :-

$$\beta_{PB} = \frac{COV_{PB}}{V_B}$$

where :-

$$\begin{aligned} COV_{PB} &= COV\left(R_P, R_B\right) \\ &= COV\left(\sum_i^N x_i R_i, \sum_j^N b_j R_j\right) \\ &= \sum_i^N \sum_j^N x_i b_j COV_{ij} \end{aligned}$$

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Portfolio Beta to the Benchmark - 3

- As an interesting alternative formulation, we have :-

$$\frac{COV(R_P, R_B)}{V_B} = COV(\sum_i^N x_i R_i, R_B)/V_B$$

$$= \sum_i^N x_i COV(R_i, R_B)/V_B$$

$$= \sum_i^N x_i \beta_{iB} = \beta_{PB}$$

- So we can also calculate the Portfolio Beta to the Benchmark from the Stock Betas in the usual way

How is this the Least Bit Useful?

- A perennial question asked by managers is about the sensitivity of Their Portfolio to some (arbitrary) Macro-Economic variable :-
  - Changes in the Oil Price
  - Changes in Inflation
  - Changes in the Term Spread

- Of course, if the Macro-Economic variable is one of the factors in the risk model, it is easy to answer

But This Doesn’t Really Work

- Macro-Economic variables don’t usually make good factors for equity risk models, since their relationship to stock returns is too weak

- On the other hand, it is often clear that **Portfolios** can have (a statistically significant) sensitivity to some Macro-Economic variables

- This can be seen by regressing the Portfolio returns against the Macro-Economic variable

A Better Methodology

- Use the fact that Macro-Economic variables do have statistically significant sensitivity to diversified portfolios

- Include the required M-E variables as assets in the Universe of Securities for some risk model

- They are then regressed on the risk model factors to give statistically significant Betas

- Then we simply set the required Macro-Economic variable as the Benchmark for Our Portfolio

Some Examples

- We used 4 different Global Sector funds
  - Energy
  - Financials
  - Health Care
  - Semi-conductors
- We also looked at the MSCI World Index
- Each of these was compared to a number of Macro-Economic variables
- Finally, we also have a real life example from a US Small cap manager running about $5billion in assets

Macro-Economic factor betas

Beta to Benchmark	Energy	Financials	Health Care	Semi-conductors	MSCI World
Crude Oil	0.256	0.020	-0.030	-0.064	0.018
Gold	0.539	0.389	0.171	0.234	0.256
USA CPI	-0.028	-1.144	-1.489	-3.094	-1.069
USA Term Spread	-0.080	0.675	1.016	3.083	0.786
Europe CPI	0.374	0.650	0.091	-0.099	0.247
Europe Term Spread	-0.225	-0.445	0.154	1.112	0.104

Energy Portfolio Risk Summary

DataItem	Portfolio	Benchmark	Relative
Factor Variance	442.96	506.60	280.00
Stock Variance	17.67	799.85	817.52
Total Variance	**460.63**	**1306.45**	**1097.52**
% Factor Variance	96.16%	38.78%	25.51%
% Stock Variance	3.84%	61.22%	74.49%
% Total Variance	**100.00%**	**100.00%**	**100.00%**
Factor Risk (s.d.)	21.05	22.51	16.73
Stock Risk (s.d.)	4.20	28.28	28.59
Total Risk (s.d.)	**21.46**	**36.14**	**33.13**

Energy Fund Relative to Oil

DataItem	Portfolio	Benchmark	Relative
Beta to Benchmark	**0.256**	**1.000**	**-0.744**
Systematic Variance	85.79	1306.45	722.68
Residual Variance	374.84	0.00	374.84
Total Variance	**460.63**	**1306.45**	**1097.52**
% Systematic Variance	18.62%	100.00%	65.85%
% Residual Variance	81.38%	0.00%	34.15%
% Total Variance	**100.00%**	**100.00%**	**100.00%**
Systematic Risk (s.d.)	9.26	36.14	26.88
Residual Risk (s.d.)	19.36	0.00	19.36
Total Risk (s.d.)	**21.46**	**36.14**	**33.13**

Financials Portfolio Risk Summary

DataItem	Portfolio	Benchmark	Relative
Factor Variance	372.92	109.37	259.78
Stock Variance	13.43	176.83	190.26
Total Variance	**386.35**	**286.20**	**450.03**
% Factor Variance	96.52%	38.22%	57.72%
% Stock Variance	3.48%	61.78%	42.28%
% Total Variance	**100.00%**	**100.00%**	**100.00%**
Factor Risk (s.d.)	19.31	10.46	16.12
Stock Risk (s.d.)	3.66	13.30	13.79
Total Risk (s.d.)	**19.66**	**16.92**	**21.21**

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Financials Fund Relative to Gold

DataItem	Portfolio	Benchmark	Relative
Beta to Benchmark	**0.389**	**1.000**	**-0.611**
Systematic Variance	43.25	286.20	106.93
Residual Variance	343.10	0.00	343.10
Total Variance	**386.35**	**286.20**	**450.03**
% Systematic Variance	11.19%	100.00%	23.76%
% Residual Variance	88.81%	0.00%	76.24%
% Total Variance	**100.00%**	**100.00%**	**100.00%**
Systematic Risk (s.d.)	6.58	16.92	10.34
Residual Risk (s.d.)	18.52	0.00	18.52
Total Risk (s.d.)	**19.66**	**16.92**	**21.21**

Macro-Economic Exposures

Example M-E Exposures	Small Cap 130-30		Russell 2500			Small Cap Fund		Russell 2500			MidCap Fund		Russell MIDCap		
Macro-Economic Variable	Portfolio Beta	Portfolio Systematic RSQ %	Benchmark Beta	Benchmark Systematic RSQ %	Relative Beta	Portfolio Beta	Portfolio Systematic RSQ %	Benchmark Beta	Benchmark Systematic RSQ %	Relative Beta	Portfolio Beta	Portfolio Systematic RSQ %	Benchmark Beta	Benchmark Systematic RSQ %	Relative Beta
Gold	-0.041	0.23	0.012	0.01	-0.053	-0.022	0.05	0.012	0.01	-0.033	-0.033	0.11	0.005	0.00	-0.038
10 Year T-Note Yield	-0.010	0.03	0.009	0.01	-0.018	-0.006	0.01	0.009	0.01	-0.015	-0.009	0.02	0.002	0.00	-0.011
Barclays AAA Muni-Bond ETF	0.075	0.98	0.098	0.80	-0.023	0.085	0.90	0.098	0.80	-0.013	0.086	0.96	0.096	0.88	-0.010
10 Year Zero Coupon T-Bond	0.297	3.42	0.476	4.15	-0.178	0.323	2.87	0.476	4.15	-0.153	0.294	2.48	0.404	3.45	-0.109
TED Spread (bps)	-0.035	5.96	-0.046	4.86	0.011	-0.040	5.50	-0.046	4.86	0.006	-0.041	5.87	-0.045	5.35	0.005
GSCI Wheat	0.099	5.77	0.145	5.80	-0.045	0.123	6.34	0.145	5.80	-0.021	0.122	6.39	0.137	5.98	-0.015
Natural Gas	0.109	9.00	0.139	6.98	-0.030	0.132	9.44	0.139	6.98	-0.007	0.135	10.16	0.136	7.70	-0.002
2 Year Yield	0.087	10.85	0.132	11.70	-0.044	0.111	12.39	0.132	11.70	-0.021	0.107	12.11	0.123	11.76	-0.016
USD Dollar Index	-0.786	18.35	-1.197	20.20	0.411	-0.922	17.99	-1.197	20.20	0.276	-0.902	17.90	-1.120	20.35	0.217
West Texas Intermediate (Oil)	0.212	22.39	0.328	25.32	-0.115	0.270	25.87	0.328	25.32	-0.058	0.266	25.93	0.310	26.17	-0.045
RICI	0.354	24.86	0.551	28.62	-0.197	0.448	28.48	0.551	28.62	-0.103	0.438	28.22	0.511	28.41	-0.073
Goldman Sachs Commodity Index	0.319	28.64	0.486	31.69	-0.168	0.403	32.71	0.486	31.69	-0.083	0.396	32.78	0.459	32.57	-0.063

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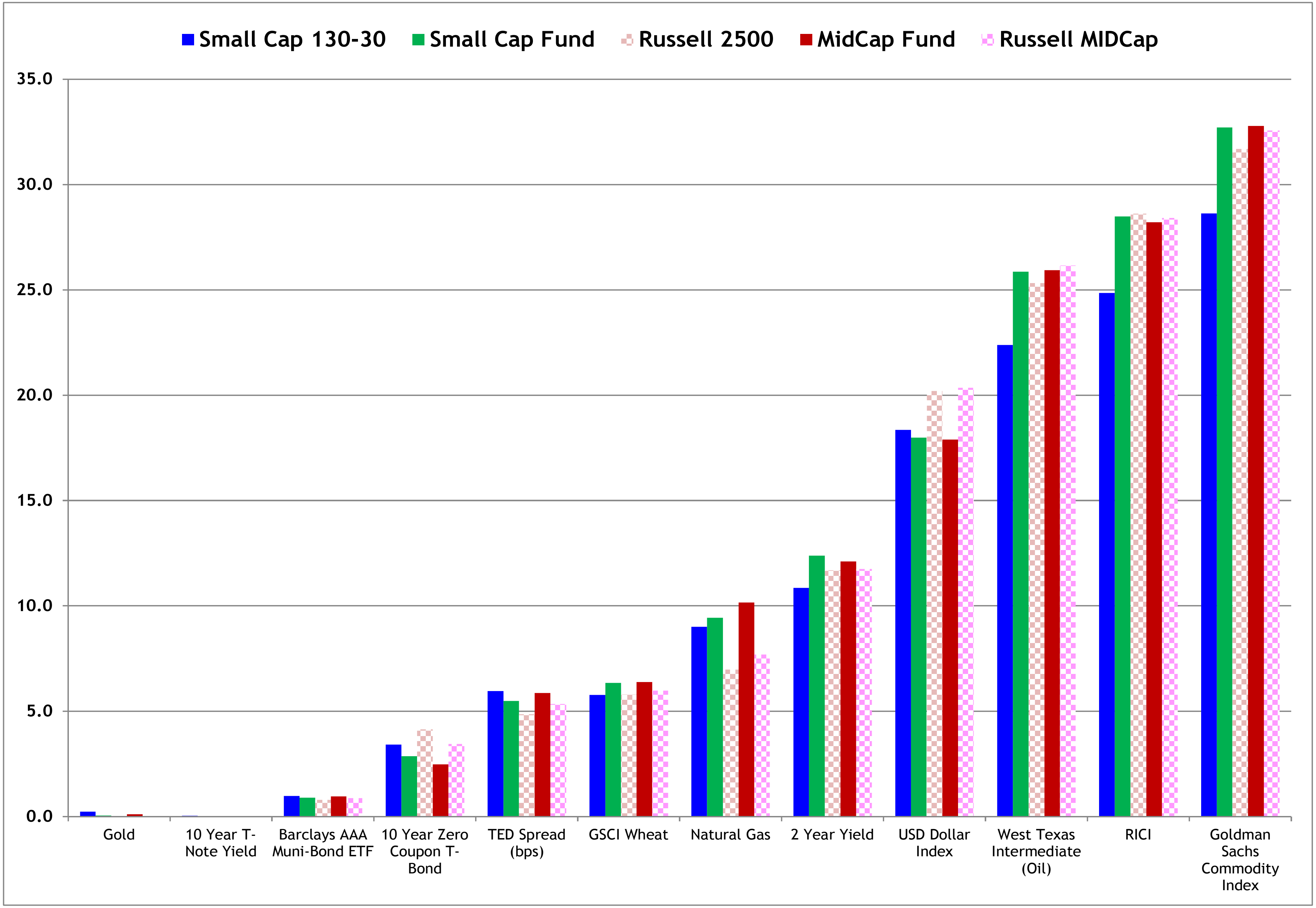
Small Cap 130-30
Small Cap Fund
Russell 2500
MidCap Fund
Russell MIDCap
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0.0
Gold
10 Year T-Note Yield
Barclays AAA Muni-Bond ETF
10 Year Zero Coupon T-Bond
TED Spread (bps)
GSCI Wheat
Natural Gas
2 Year Yield
USD Dollar Index
West Texas Intermediate (Oil)
RICI
Goldman Sachs Commodity Index

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Summarising the ‘Some Benchmark’ Model

- Vendors of risk models routinely provide the Beta of a Portfolio to its Benchmark, despite this being somewhat incongruous

- While this may be mildly interesting, it is only a particular case of a much more general application, which can provide the sensitivity of a portfolio to any Macro-Economic variables

3. The 'Your Portfolio' Model

- In this model, **Your Portfolio** is the single factor, and we model the returns to individual stocks by their relationship with the whole portfolio

- The model, which is structurally similar to the Market Model, is as follows :-

$$R_{it} = \beta_{iP} R_{Pt} + \alpha_{it}$$

- However, to see why it is useful, we will approach by a somewhat different route …

Portfolio Return and Risk

- Portfolio Return is defined as :-

$$R_{Pt} = \sum_{i}^{N} x_i R_{it}$$

- From which we derive Portfolio Risk as follows, where COV_{ij} is a full covariance matrix :-

$$V_P = \sum_{i}^{N} \sum_{j}^{N} x_i x_j COV_{ij}$$

Portfolio Risk Decomposition

- We should be interested in how much risk comes from each holding, or each group of holdings (e.g. all the Energy stocks or all the UK stocks)
- We first define the individual stock contributions to portfolio variance as :-

$$ACV_{iP} = \sum_{j}^{N} x_i x_j COV_{ij}$$

A Little Bit of Simplifying Algebra

$$ACV_{iP} = \sum_{j}^{N} x_i x_j COV_{ij} \quad = x_i \sum_{j}^{N} x_j COV_{ij}$$

$$= x_i \sum_{j}^{N} x_j COV\left(R_i, R_j\right) \quad = x_i COV(R_i, \sum_{j}^{N} x_j R_j)$$

$$= x_i COV\left(R_i, R_P\right) \quad = x_i COV_{iP}$$

where COV_{iP} is the covariance of stock i with the whole portfolio P.

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Percentage Risk Contribution

- Since no-one can understand variances, it helps to convert Actual Contributions to Variance to Percentage Contributions :

$$PCV_{iP}\% = 100\frac{ACV_{iP}}{V_P} = 100\frac{x_i COV_{iP}}{V_P}$$

- We simply divide the Actual Contribution by the Total Variance and multiply by 100

The Beta of a Stock to the Portfolio

- As before, the beta of a stock ***i*** to the portfolio ***P*** in our model is given by :-

$$\beta_{iP} = \frac{COV_{iP}}{V_P}$$

- Using this, we see that the Percent Contribution to Variance becomes :-

$$PCV_{iP} = 100 x_i \frac{COV_{iP}}{V_P} = Pct_i \beta_{iP}$$

Practical Applications

- Perhaps more usefully, we can write :-

$$\beta_{iP} = \frac{PCV_{iP}}{Pct_i}$$

- This ratio of the percent of risk to the percent holding size is called the Relative Imbalance

- It immediately tells the manager whether each holding in the portfolio is more or less risky than average (which is 100%/100% = 1 = β_{PP} of course)

Contributions from Groups of Holdings

- We can generalise these expressions from individual holdings to groups of holdings as follows :-

$$ACV_{Energy} = \sum_{i \in Energy} ACV_{iP}$$

$$PCV_{Energy}\% = \sum_{i \in Energy} PCV_{iP}\%$$

Beta of a Group of Holdings

- From this, it is trivial to observe that we can then calculate the Beta of a group to the Portfolio in the same way :-

$$\beta_{EnergyP} = \frac{PCV_{EnergyP}}{Pct_{Energy}}$$

- Which tells us whether our bet on Energy is more or less than average for the portfolio

Marginal Contribution to Variance

- For the total portfolio risk we have :-

$$V_P = \sum_i^N \sum_j^N x_i x_j COV_{ij}$$

- The Marginal Contribution to Variance is then defined as :-

$$MCV_{iP} = \frac{\partial V_P}{\partial x_i} = 2\sum_j^N x_j COV_{ij} = 2COV_{iP}$$

  - which really couldn't be simpler!

Marginal Contribution to Risk (Standard Deviation)

- Bearing in mind that : $V_p = S_p^2$

  we have : $\dfrac{\partial V_P}{\partial S_P} = 2S_P$

- And so, trivially,

$$MCR_{iP} = \frac{\partial S_P}{\partial x_i} = \frac{\partial S_P}{\partial V_P}\frac{\partial V_P}{\partial x_i} = \frac{\dfrac{\partial V_P}{\partial x_i}}{\dfrac{\partial V_P}{\partial S_P}} = \frac{MCV_{iP}}{2S_P}$$

Deriving MCR_{iP} from β_{iP}

- A little further work shows us that the Marginal Contribution to Risk of stock i is simply its Beta to the Portfolio multiplied by the Portfolio Risk

$$MCR_{iP} = \frac{MCV_{iP}}{2S_P} = \frac{2COV_{iP}}{2S_P} = \frac{COV_{iP}}{S_P} = \beta_{iP} S_P$$

- Finally, we can also use our new friend β_{iP} to determine the efficiency of each holding in the portfolio …

Reverse Optimisation

- A full derivation of this algorithm is beyond the scope of this presentation

- However, the main result is as follows :-

$$I(R_i) = E(R_P) + \psi S_P(\beta_{iP} - 1)$$

- This gives the returns $I(R_i)$ required for each stock to make the portfolio efficient, given some value of the risk aversion parameter, ψ .

The Formal Result

- This equation states that, for efficiency, the expected return on a stock should be equal to the Expected Portfolio return plus an adjustment

- This expected return adjustment will be positive or negative, depending on whether the particular stock is more risky $(\beta_{iP} > 1)$ or less risky $(\beta_{iP} < 1)$ than average

- Note that none of the other values in this formula relate to individual stocks

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Risk Aversion Parameter

- The actual size of the adjustment also depends on ψ , the incremental risk aversion parameter
- The more risk averse the portfolio manager is, the larger the value of ψ, and hence the higher the returns required on the more risky holdings (and the lower the returns required on the less risky holdings)
- For rational investors, of course, $\psi > 0$

A Particular Case

- Consider the case where the risk aversion parameter, or incremental return/risk trade-off is the same as the overall portfolio return/risk trade-off

Substituting $\psi = E\left(R_P\right)/S_P$ we get :-

$$I\left(R_i\right) = E\left(R_P\right) + \frac{E(R_P)}{S_P} S_P\left(\beta_{iP} - 1\right)$$

$$= E\left(R_P\right) + E(R_P)\left(\beta_{iP} - 1\right)$$

So : $I\left(R_i\right) = \beta_{iP} E(R_P)$

Practical Applications - 1

- Ranking the stocks held in a portfolio from high to low by their Beta immediately gives us the implied ranking of assets by their relative attractiveness
- This is particularly useful for portfolio managers who are unable (or unwilling) to quantify their return expectations
- For such managers, portfolio **inefficiency** consists of having their favourite stocks too far down the list, and their no-so-favourite stocks being near the top
- This analysis usually suggests some obvious pairs trades to improve the overall portfolio efficiency

Practical Applications - 2

- For managers brave enough to quantify their return expectations, there are two obvious applications
- The Portfolio Return $E(R_p)$ can be generated from the manager's own expectations, and given S_p we can then generate Implied Returns for any value of ψ.
- Alternatively, we can use this parameter to adjust the scale of the Implied Returns so that they are on the same scale as the actual Expected Returns, and then compare them directly

Summarising the Your Portfolio Model - 1

- We have introduced the notion of the Beta of a stock to Your Portfolio

- These Betas can tell us a lot about the risk structure of the portfolio, and also what stock returns are required for efficiency

- Note that, in the Your Portfolio Model, Systematic means Portfolio-related and Residual means Portfolio-independent

Summarising the Your Portfolio Model - 2

- For each stock held in Your Portfolio, we have :-

Beta of Stock to Your Portfolio : $\beta_{iP} = COV_{iP} / V_P$

Actual Contribution to Variance : $ACV_{iP} = \beta_{iP} x_i V_P$

Actual Contribution to Risk : $ACS_{iP} = \beta_{iP} x_i S_P$

Percent Contribution to Variance : $PCV_{iP} = \beta_{iP} Pct_i$

Percent Contribution to Risk : $PCS_{iP} = \beta_{iP} Pct_i$

Summarising the Your Portfolio Model - 3

- For each stock held in Your Portfolio, we have :-

Marginal Contribution to Risk : $MCR_{iP} = \beta_{iP} S_P$

Implied Return for Efficiency (general case) :

$$I(R_i) = E(R_P) + \psi S_P (\beta_{iP} - 1)$$

Implied Return when $\psi = E(R_P) / S_P$

is given by : $I(R_i) = \beta_{iP} E(R_P)$

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Polyphemus' Perspective

- Single factor models have only ever been used in Finance for theoretical purposes
- However, managers often have questions about some form of Systematic/Residual split in the risks of their portfolios
- A judiciously-chosen single-factor risk model can provide useful insights into the risk structure and efficiency of a portfolio

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